

Gravitational potential of a homogeneous circular torus: new approach

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ABSTRACT

The integral expression for gravitational potential of a homogeneous circular torus composed of infinitely thin rings is obtained. Approximate expressions for torus potential in the outer and inner regions are found. In the outer region a torus potential is shown to be approximately equal to that of an infinitely thin ring of the same mass; it is valid up to the surface of the torus. It is shown in a first approximation, that the inner potential of the torus (inside a torus body) is a quadratic function of coordinates. The method of sewing together the inner and outer potentials is proposed. This method provided a continuous approximate solution for the potential and its derivatives, working throughout the region.

Key words: galaxies: general - gravitation: gravitational potential - torus

1 INTRODUCTION

Toroidal structures are now detected in astrophysical objects of various types. Such objects are, for example, the ring galaxies, where a ring of stars is observed. In some galaxies, the ring-like distribution of stars is believed to be due to collisions of galaxies, as, for example, in M31 (Block et al. 1987), and Arp 147 (Gerber et al. 1992). The analysis of the SDSS data (Ibata et al. 2003) indicates the existence of a star ring in the Milky Way on scales of about 15–20 kpc, which is believed to be originated from the capture of a dwarf galaxy. Obscuring tori are observed in central regions of active galactic nuclei (AGN) (Jaffe et al. 2004) and play an essential role in the unified scheme (Antonucci 1993; Urry & Padovani 1995). Ring-like structures exhibit themselves in dark matter as well. An example can be the galaxy cluster C10024+17 where a ring-like structure has been found in distribution of dark matter with the use of gravitational lensing method (Jee et al. 2007). In the Milky Way, the rotation curves together with the EGRET data can be explained by existence of two rings of dark matter located at distances of about 4 kpc and 14 kpc from the Galaxy center (de Boer et al. 2005). Such toroidal structures can possess a significant mass, and thus gravitationally affect the matter motion.

B. Riemann devoted one of his last works to the gravitational potential of a homogeneous torus (see Collected papers, 1948). This work remained unfinished however. For over a century, no attention has been paid to a torus gravi-

tational potential¹. Kondratyev (2003) has returned to this problem for the first time. In this work an exact expression for the potential of a homogeneous torus on the axis of symmetry was obtained. In (Kondratyev 2007) the integral expressions for a homogeneous torus potential were found using a disk as a primordial gravitating element. Stacking up such disks will result in a torus with the potential equal to a sum of potentials of component disks. However, it is evident that any integral expressions are problematic to use both in analytic studies and in numerical integration of motion equations, and also in solving the problems of gravitational lensing. B.Kondratyev et al. (2009, 2010) have obtained an expansion of torus potential in Laplace series, but showed, however, that such an expansion is impossible inside some spherical shell.

In this paper we propose a new approach to investigation of the gravitational potential of a torus. Special attention has been paid to finding approximate expressions for the potential, which would simplify investigation of astrophysical objects with gravitating tori as structural elements. In contrast to (Kondratyev 2007), we used an infinitely thin ring as a torus component. Such ring is actually a realization of a torus, with its minor radius tending to zero and the major one equaling the ring radius. Using such an approach, we obtained an integral expression for the potential

¹ In electrostatics, a potential of conducting torus shell is used (Smythe 1950) that is much easier than the case when torus density is uniform inside its volume.

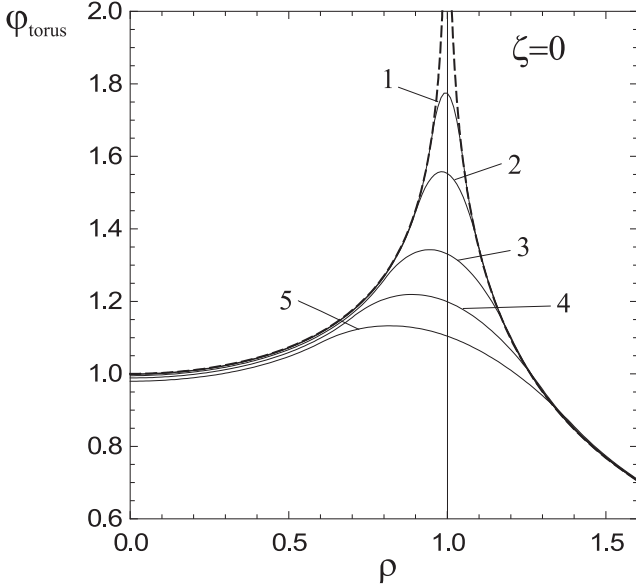


Figure 2. Dependence of the potential on the radial coordinate for $\zeta = 0$, for the tori with various values of geometrical parameter: 1) $r_0 = 0.05$, 2) $r_0 = 0.1$, 3) $r_0 = 0.2$, 4) $r_0 = 0.3$, 5) $r_0 = 0.4$. The potential of an infinitely thin ring (1) with mass M equal to the mass of the torus is shown by the dashed line. In this plot and in all the subsequent figures, $M = 1$, $R = 1$, $G = 1$.

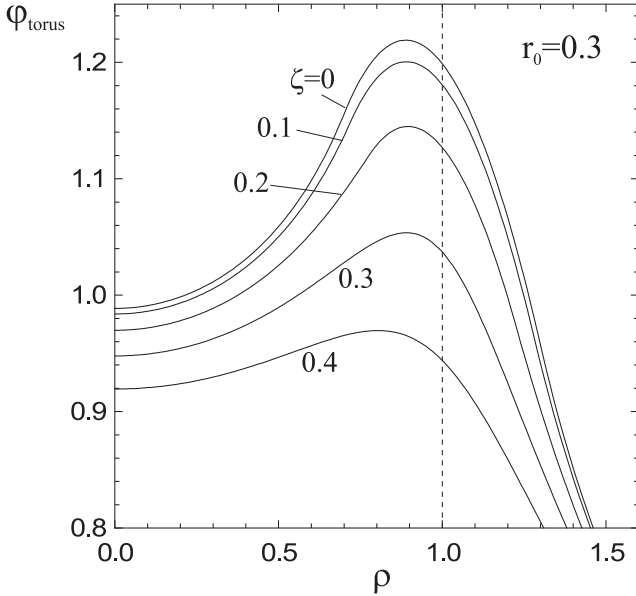


Figure 3. Potential of the torus with $r_0 = 0.3$ as a function of the radial coordinate for various values of $\zeta = 0, 0.1, 0.2, 0.3, 0.4$.

the torus potential on the radial coordinate are presented, which were calculated from expression (8) for different values of ζ .

Note, that in contrast to the work by Kondratyev (2007, p.196, expression (7.26)), where the torus potential is expressed only through a single integration of the elliptical integrals of all the three kinds, the torus potential (8) in our work is expressed by double integration of the elliptical integral of the first kind. However, further analysis of this expression for the torus potential (8) allows us to ob-

tain approximations that are physically understandable and enable solving practical astrophysical tasks which need multiple calculations of the gravitational potential of the torus.

For further analysis of the torus potential, we define the inner region as the volume bounded by the torus surface (inside the torus body) and outer region as the region outside this surface.

3 TORUS POTENTIAL IN THE OUTER REGION

It is seen from Fig. 2 that the outer potential of the torus can be approximately represented by the potential of an infinitely thin ring of the same mass up to torus surface. For $\rho \rightarrow 0$, the values of the torus potential and potential of the infinitely thin ring differ by a quantity that depends on a geometric parameter r_0 , that is especially evident for a thick torus ($r_0 > 0.5$). Find a relationship between the outer potential of the torus and the potential of a ring of the same mass, that is, derive an approximate expression for torus potential in the outer region, where a condition $(\rho - 1)^2 + \zeta^2 \geq r_0^2$ holds. Within this region, the integrand $\phi_r(\rho, \zeta; \eta', \zeta')$ in (8) does not have singularities for all η', ζ' , therefore, it can be expanded as the Maclaurin series in powers of η', ζ' in the vicinity of a point $\eta' = \zeta' = 0$. Since the integrals in symmetrical limits from the series terms that contain cross derivatives and derivatives of the odd orders are equal to zero, only summands with the even orders remain in the expansion. With the quadratic terms of the series being restricted, the potential of the component ring is:

$$\phi_r(\rho, \zeta; \eta', \zeta') \approx \phi_c(\rho, \zeta) + \frac{1}{2} \frac{\partial^2 \phi_r}{\partial \eta'^2} \Big|_{\eta'=0, \zeta'=0} \eta'^2 + \frac{1}{2} \frac{\partial^2 \phi_r}{\partial \zeta'^2} \Big|_{\eta'=0, \zeta'=0} \zeta'^2. \quad (9)$$

Substituting (9) into (8), we will have after integration:

$$\varphi_{\text{torus}} \approx \frac{GM}{\pi R} \phi_c \cdot \left(1 + \frac{r_0^2}{2\phi_c} \left[\frac{\partial^2 \phi_r}{\partial \eta'^2} \Big|_{\eta'=0, \zeta'=0} + \frac{\partial^2 \phi_r}{\partial \zeta'^2} \Big|_{\eta'=0, \zeta'=0} \right] \right). \quad (10)$$

Ultimately, the approximate expression for the torus potential in the outer region $(\rho - 1)^2 + \zeta^2 \geq r_0^2$ has a form:

$$\varphi_{\text{torus}}(\rho, \zeta; r_0) \approx \frac{GM}{\pi R} \phi_c \cdot \left(1 - \frac{r_0^2}{16} + \frac{r_0^2}{16} \cdot S(\rho, \zeta) \right), \quad (11)$$

where $\phi_c = \sqrt{\frac{m}{\rho}} K(m)$ is a dimensionless potential of the central ring (2), and

$$S(\rho, \zeta) = \frac{\rho^2 + \zeta^2 - 1}{(\rho + 1)^2 + \zeta^2} \cdot \frac{E(m)}{K(m)}, \quad (12)$$

$E(m) = \int_0^{\pi/2} d\beta \sqrt{1 - m \sin^2 \beta}$ is the complete elliptical integral of the second kind. We may conveniently proceed to a new variable $\eta = \rho - 1$ that allows expression (12) to be represented as

$$S(\eta, \zeta) = \frac{\eta^2 + \zeta^2 + 2\eta}{(\eta + 2)^2 + \zeta^2} \cdot \frac{E(m)}{K(m)}, \quad (13)$$

where

$$m = 4 \frac{\eta + 1}{(\eta + 2)^2 + \zeta^2}.$$

Expression (11) for the torus potential (we will further call it the S -approximation), with (12) or (13) taken into account, represents the torus potential accurately enough in the outer region $\eta^2 + \zeta^2 \geq r_0^2$ (Fig. 2). Since $|S| \leq 1$ the second multiplier in (11) is a slowly varying function in ρ and ζ . Let us simplify the expression (11) replacing the second multiplier by its asymptotic approximations.

In the first case, $\rho \rightarrow 0$ corresponding to $\eta \rightarrow -1$, the parameter $m \rightarrow 0$ and $E(m)/K(m) \rightarrow 1$, therefore, $S \rightarrow (\zeta^2 - 1)/(\zeta^2 + 1)$. The expression for the torus potential in this case is

$$\varphi_{\text{torus}}(\rho, \zeta; r_0) \approx \frac{GM}{\pi R} \phi_c(\rho, \zeta) \cdot \left(1 - \frac{r_0^2}{16} + \frac{r_0^2}{16} \frac{\zeta^2 - 1}{\zeta^2 + 1}\right). \quad (14)$$

Since the dimensionless potential of the ring (2) at the symmetry axis is $\phi_c = \frac{\pi}{\sqrt{1+\zeta^2}}$, we get for the torus

$$\varphi_{\text{torus}}(0, \zeta; r_0) \approx \frac{GM}{R} \frac{1}{\sqrt{1+\zeta^2}} \cdot \left(1 - \frac{r_0^2}{16} + \frac{r_0^2}{16} \frac{\zeta^2 - 1}{\zeta^2 + 1}\right). \quad (15)$$

and for $\zeta = 0$,

$$\varphi_{\text{torus}}(0, 0; r_0) \approx \frac{GM}{R} \left(1 - \frac{r_0^2}{8}\right). \quad (16)$$

The second **summand** $GM/R \cdot r_0^2/8$ in (16) describes displacement of the torus potential at the symmetry axis as compared to the potential of an infinitely thin ring (Fig. 2).

In the second case, at large η , the parameter $m \rightarrow 0$, and $S \rightarrow 1$ in (11), therefore:

$$\varphi_{\text{torus}}(\rho, \zeta; r_0) \approx \frac{GM}{\pi R} \phi_c(\rho, \zeta), \quad (17)$$

that is, the torus potential is equal to the potential of the infinitely thin ring with the same mass M and radius R in this case. It is seen from Fig. 4 that the S -approximation for the torus outer potential (11) is applicable up to the torus surface (upper curves). Indeed, in the region $\rho \leq 1 - r_0$, difference between the potential obtained from the integral expression (8) and its value taken from the S -approximation reaches maximum near the torus surface and does not exceed 0.2% for $r_0 = 0.5$. The difference remains small even for a thick torus: it does not exceed 1.5% for $r_0 = 0.9$. For $\zeta = r_0$ all the points are outer, and the curves for the exact potential and S -approximation virtually coincide (deviation is less than 0.1%).

Note that asymptotics of the S -approximation for the outer potential (15) and (17) also describe the torus potential well enough (dotted line in Fig. 4). Thus, for $|\zeta| < r_0$, the approximation (15) can be used to estimate the potential inside the region bounded by a cylinder with radius $\rho - r_0$, while the approximation (17) is applicable outside the region bounded by a cylinder with radius $\rho + r_0$. At $|\zeta| \gg 1$, expression (15) tends to (17), and expression for potential of the infinitely thin ring (2) can be used within the whole outer region to approximately evaluate the torus potential.

Therefore, *the outer potential of the torus can be represented with good accuracy by a potential of an infinitely thin ring of the same mass. The dependence of the geometrical parameter r_0 appears only in the torus hole; it is taken into account in the "shifted" potential of the infinitely thin ring*

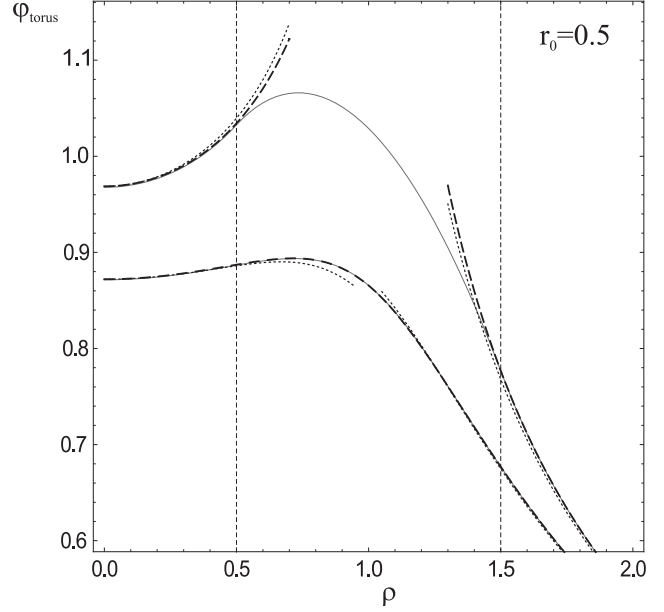


Figure 4. Dependence of the torus potential with $r_0 = 0.5$ on ρ for $\zeta = 0$ (upper curves) and $\zeta = 0.5$ (lower curves). Solid lines show the potential calculated with the exact formula (8). The S -approximation (11) of the potential is shown in dashed lines, and dotted lines represent the limiting cases of the S -approximation: the potential curve for the infinitely thin ring (17) is to the right of the torus cross-section, and the curve representing a "shifted" potential of the infinitely thin ring (14) is to the left. The boundaries of the torus cross-section are dotted with the vertical lines.

(14). These approximations are valid up to the surface of the torus.³

4 TORUS POTENTIAL IN THE INNER REGION

To analyze the inner potential of the torus, it is convenient to select the origin of a coordinate system in the center of the torus cross-section (Fig. 5). Then, the dimensionless potential of the central ring takes a form:

$$\phi_c(\eta, \zeta) = \sqrt{\frac{m}{1+\eta}} \cdot K(m) \quad (18)$$

where

$$m = \frac{4(1+\eta)}{(2+\eta)^2 + \zeta^2}.$$

Consider the potential of the central ring (18) in the vicinity of $\eta \rightarrow 0$, $\zeta \rightarrow 0$, that corresponds to $m \rightarrow 1$. In this case, the elliptical integral in (18) can be expanded in terms of a small parameter $m_1 = 1 - m$. With the series clipped by two terms, we will have:

$$K(m_1) \approx \ln \frac{4}{\sqrt{m_1}} + \frac{1}{4} m_1 \ln \frac{4}{e\sqrt{m_1}}, \quad (19)$$

³ There is some analogy with the known result: the outer potential of a solid sphere of mass M is the same as that generated by a point mass M located at the sphere's center. Note, however, that torus has another system of equigravitating elements (Kondratyev, 2007).

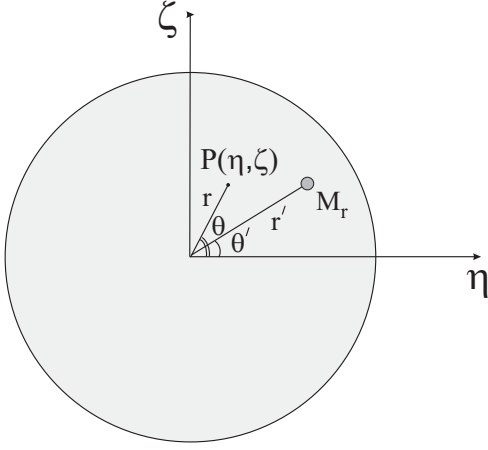


Figure 5. The scheme of torus cross-section.

where

$$m_1 = \frac{r^2}{r^2 + 4(1 + \eta)},$$

$r^2 = \eta^2 + \zeta^2$. The approximate formula for the ring potential expressed through the parameter m_1 is then:

$$\phi_c(\eta, \zeta) \approx \frac{2\sqrt{m_1}}{r} \left(\ln \frac{4}{\sqrt{m_1}} + \frac{1}{4} m_1 \ln \frac{4}{e\sqrt{m_1}} \right). \quad (20)$$

Passage to the potential of an arbitrary component ring is fulfilled by substitutions $1 + \eta \rightarrow (1 + \eta)/(1 + \eta')$ and $\zeta \rightarrow (\zeta - \zeta')/(1 + \eta')$, which results in an expression

$$\phi_r(\eta, \zeta; \eta', \zeta') \approx \frac{2(1 + \eta')}{\sqrt{q}} \left[\ln \frac{4}{\sqrt{m'_1}} \left(1 + \frac{m'_1}{4} \right) - \frac{m'_1}{4} \right] \quad (21)$$

where $m'_1 = (\mathbf{r} - \mathbf{r}')^2/q$, $q = (\mathbf{r} - \mathbf{r}')^2 + 4(1 + \eta)(1 + \eta')$. A summand $(\mathbf{r} - \mathbf{r}')^2 = (\eta - \eta')^2 + (\zeta - \zeta')^2$ is a square of the distance between the component ring and a point P (Fig. 5). Expansion (21) is valid for $m'_1 \rightarrow 0$, therefore, $(\mathbf{r} - \mathbf{r}')^2 \ll 1$.

Confine ourselves by the case of a thin torus ($r_0 \ll 1$). Then $(\mathbf{r} - \mathbf{r}')^2 \ll 4(1 + \eta)(1 + \eta')$ and $(1 + \eta)(1 + \eta') \approx 1$, and the first multiplier in (21) can be written to the second-order terms as:

$$f_1 \equiv \frac{2(1 + \eta')}{\sqrt{q}} \approx \sqrt{\frac{1 + \eta'}{1 + \eta}} \left(1 - \frac{1}{8}(\mathbf{r} - \mathbf{r}')^2 \right). \quad (22)$$

After expanding the square root in (22) in powers of η and η' , we obtain

$$f_1 \approx \left(1 + \frac{\eta'}{2} - \frac{\eta}{2} - \frac{\eta\eta'}{4} - \frac{\eta'^2}{8} + \frac{3}{8}\eta^2 \right) \left(1 - \frac{1}{8}(\mathbf{r} - \mathbf{r}')^2 \right). \quad (23)$$

Similarly, the second multiplier (in square brackets) in expression (21) can be written to the terms quadratic in coordinates as:

$$f_2 \equiv \ln \frac{4}{\sqrt{m'_1}} \left(1 + \frac{m'_1}{4} \right) - \frac{m'_1}{4} \approx \frac{1}{2}(\eta + \eta') - \frac{1}{4}(\eta^2 + \eta'^2) + \ln \frac{8}{|\mathbf{r} - \mathbf{r}'|} + \frac{(\mathbf{r} - \mathbf{r}')^2}{16} \ln \frac{8e}{|\mathbf{r} - \mathbf{r}'|}. \quad (24)$$

Thus, we obtain the following expression for the potential

of the component ring:

$$\phi_r(\eta, \zeta; \eta', \zeta') \approx f_1 \cdot f_2. \quad (25)$$

In consideration of the inner potential of the torus, rewrite expression (8) in the polar coordinates (Fig. 5):

$$\varphi_{torus}(r, \theta; r_0) = \frac{GM}{\pi^2 R r_0^2} \int_0^{r_0} \int_0^{2\pi} \phi_r(r, \theta; r', \theta') r' dr' d\theta', \quad (26)$$

where coordinates of the component ring are $\eta' = r' \cos \theta'$, $\zeta' = r' \sin \theta'$, and coordinates of a point P are $\eta = r \cos \theta$, $\zeta = r \sin \theta$. Substitute (23) and (24) into (25), and after multiplying, restrict ourselves by the terms quadratic in η , ζ and η' , ζ' . Then, after integration of (26), we obtain the approximate expression for the inner potential of the torus:

$$\varphi_{torus}(\eta, \zeta; r_0) \approx \frac{GM}{2\pi R} \left[c + \tilde{a}_1 \eta + \tilde{a}_2 \eta^2 + \tilde{b}_2 \zeta^2 \right], \quad (27)$$

where

$$k \equiv \frac{r_0}{8}, \quad c = 1 + 2k^2 - 2 \ln k + 8k^2 \ln k, \quad \tilde{a}_1 = 1 + \ln k,$$

$$\tilde{a}_2 = -\frac{1}{(8k)^2} - 4k^2(11 + 10 \ln k),$$

$$\tilde{b}_2 = -\frac{1}{(8k)^2} + 4k^2(3 + 2 \ln k).$$

The first summand in (27) is the value of the torus potential in the center of the torus cross-section: $c = \phi_{torus}(0, 0; r_0)$. To further analyze the inner potential, it is convenient to transfer to a coordinate system normalized to the geometrical parameter of the torus r_0 . Then the series coefficients will transform to the form:

$$a_1 = 8k(1 + \ln k), \quad a_2 = -1 - 4k^2(11 + 10 \ln k),$$

$$b_2 = -1 + 4k^2(3 + 2 \ln k).$$

The expression for the torus potential (27) can be written as

$$\varphi_{torus}(\eta, \zeta; r_0) \approx \frac{GM}{2\pi R} \left[c + a_1 \frac{\eta}{r_0} + a_2 \left(\frac{\eta}{r_0} \right)^2 + b_2 \left(\frac{\zeta}{r_0} \right)^2 \right]. \quad (28)$$

It follows from (28) that the maximal value of the potential reaches at a point $\eta_{max} = -(a_1 r_0)/(2a_2)$, $\zeta = 0$, while equipotential lines are ellipses with their centers displaced an amount η_{max} with respect to the center of the torus cross-section, and a ratio of semiaxes of the ellipses is $\sqrt{b_2/a_2}$. Note, that location of the potential maximum $\eta = \eta_{max}$, $\zeta = 0$ corresponds to the weightlessness point, where the resultant of all the forces affecting a particle inside the torus equals zero. In such an approximation, components of the force inside the torus depend on the coordinates linearly at that. In Fig. 6, the curves of the inner potential in the coordinate system normalized to r_0 are presented for three values of r_0 .

Though we confined ourselves to the case of a thin torus, the curves of potential taken from expression (28) are well consistent with the curves for the exact potential (8) up to $r_0 = 0.5$, where the deviation is maximal near the torus surface and is of the order of 2% (Fig. 6). The value of potential in the center of the torus cross-section (a constant

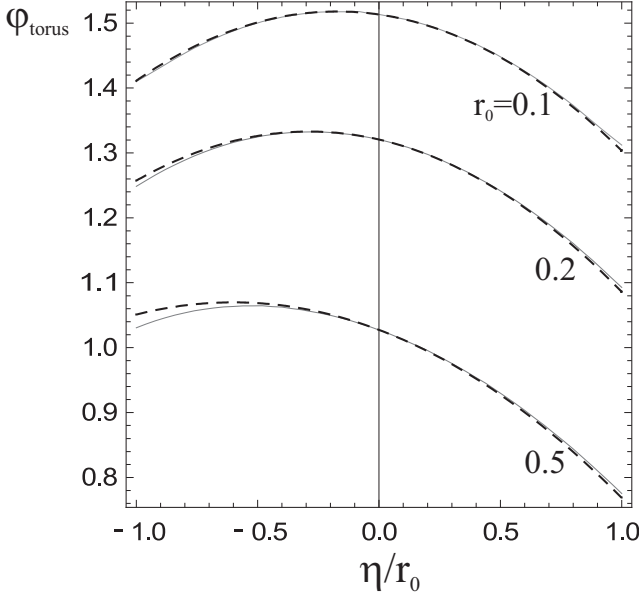


Figure 6. Dependence of the inner potential on the normalized coordinate η/r_0 at $\zeta = 0$ for various values of the geometric parameter: $r_0 = 0.1, 0.2, 0.5$. Solid curves represent dependencies of the potential on the distance to the center of the torus cross-section, which were calculated with the exact formula (8). Dashed lines are the potential curves taken with the approximate expression for the inner potential of the torus (28).

$c = \phi_{torus}(0, 0; r_0)$ in (27)), also coincides with its exact value.

It is of interest to investigate the solutions obtained for limiting cases. Indeed, the case $r_0 = R_0/R \rightarrow 0$ corresponds to two limiting passages to an infinitely thin ring (R is fixed while $R_0 \rightarrow 0$) and to a cylinder, when R_0 is fixed and $R \rightarrow \infty$. Dwell on the limiting passage to the cylinder potential. At $r_0 \rightarrow 0$, the coefficients $c \rightarrow 1 - 2 \ln k$, $a_1 \rightarrow 0$, and coefficients $a_2, b_2 \rightarrow -1$, and expression (28) takes the following form in this case:

$$\varphi_{torus}(\eta, \zeta; r_0 \rightarrow 0) \approx \frac{GM}{2\pi R} \left[1 - 2 \ln \frac{r_0}{8} - \left(\frac{r}{r_0} \right)^2 \right] \quad (29)$$

where $r^2 = \eta^2 + \zeta^2$. It is known that the inner potential of a circular cylinder with the length $2H$, much larger than the radius R_0 of its cross-section (Kondratyev 2007), has the form:

$$\varphi_{cyl} = \frac{GM}{2H} \left[2 \ln \frac{2\sqrt{e}H}{R_0} - \left(\frac{r}{r_0} \right)^2 \right]. \quad (30)$$

After a formal substitution $2H = 2\pi R$ in (30), (the cylinder length is equal to the length of the central ring), we get an expression:

$$\varphi_{cyl} = \frac{GM}{2\pi R} \left[1 - 2 \ln \left(\frac{r_0}{2\pi} \right) - \left(\frac{r}{r_0} \right)^2 \right]. \quad (31)$$

Expression (31) coincides with (29) to a constant ⁴.

The quadratic dependence on r for the inner potential

⁴ Difference in the constant may be caused by the curvature of the torus surface.

of a thin torus can be also derived in the case, when the minor radius $R_0 \rightarrow 0$. The outer potential of the torus was shown in section 3 to be approximately equal to the potential of an infinitely thin ring of the same mass and radius. In this case, the smaller is the torus geometrical parameter r_0 , the more accurate is this approximation. Therefore, at $\eta^2 + \zeta^2 \geq r_0 \rightarrow 0$, the outer potential of the torus tends to the potential of an infinitely thin ring. In this case, $\eta, \zeta \rightarrow 0$, and thus, the elliptical integral in the expression for an infinitely thin ring (2) can be expanded in the vicinity of $m \rightarrow 1$. If we confine ourselves to the first term of the expansion, we get an approximate expression for the potential of a central infinitely thin ring

$$\varphi_c(\eta, \zeta) \approx \frac{GM}{2\pi R} (-\ln(\eta^2 + \zeta^2) + 2 \ln 8), \quad (32)$$

which remains valid for the outer potential of the thin torus as well. It should be noted that there is no dependence on r_0 , because in such an approximation, all thin tori with the same masses and major radii are equigravitating for the outer potential. The derivatives of the outer potential of the thin torus in η, ζ are then:

$$\frac{\partial \varphi_c}{\partial \eta} \approx -\frac{GM}{\pi R} \frac{\eta}{r^2}, \quad \frac{\partial \varphi_c}{\partial \zeta} \approx -\frac{GM}{\pi R} \frac{\zeta}{r^2}$$

and take the following forms at the torus surface ($\eta^2 + \zeta^2 = r_0^2$):

$$\left. \frac{\partial \varphi_c}{\partial \eta} \right|_{r=r_0} \approx -\frac{GM}{\pi R r_0} \cos \theta, \quad \left. \frac{\partial \varphi_c}{\partial \zeta} \right|_{r=r_0} \approx -\frac{GM}{\pi R r_0} \sin \theta$$

It is the linear dependence of the force on coordinates η, ζ that satisfies such boundary conditions. Thus, the inner potential of the thin torus can be represented to the integration constant in the form:

$$\varphi_{torus}(\eta, \zeta; r_0) \approx \frac{GM}{2\pi R} \left(-\frac{\eta^2 + \zeta^2}{r_0^2} + c(r_0) \right). \quad (33)$$

Equating (32) with (33) at the torus surface, we obtain the expression for the constant $c(r_0) = -2 \ln(r_0/8) + 1$ that coincides with expression (27) obtained above at $r_0 \ll 1$.

It becomes evident from analysis of the inner potential for the two limiting cases ($R_0 \rightarrow 0$ and $R \rightarrow \infty$) that the first summand in coefficients a_2, b_2 of the power series (28) represents properties of the inner potential of a cylinder. With the cylinder potential separated, the inner potential of the torus (28) can be written as:

$$\varphi_{torus}(\eta, \zeta) = \varphi_{cyl}(r) + \varphi_{curv}(\eta, \zeta), \quad (34)$$

where

$$\varphi_{curv} \approx \frac{GM}{2\pi R} \left[c_{curv} + a_1 \left(\frac{\eta}{r_0} \right) + c_a \left(\frac{\eta}{r_0} \right)^2 + c_b \left(\frac{\zeta}{r_0} \right)^2 \right], \quad (35)$$

$$c_{curv} = 2 \ln \frac{8}{2\pi} + 2k^2(1 + 4 \ln k),$$

$$c_a = 1 + a_2, \quad c_b = 1 + b_2.$$

The second summand $\varphi_{curv}(\eta, \zeta)$, which we will call a potential of curvature, implies curvature of the torus surface. Indeed, all the coefficients of the series (35) tend to zero in the limiting passage to the cylinder ($r_0 \rightarrow 0$), and $\varphi_{curv} \rightarrow 0$. Therefore, the inner potential of the torus can be represented

as a sum of the cylinder potential and a term comprising a geometrical curvature of the torus surface.

5 SEWING TOGETHER THE INNER AND OUTER POTENTIALS AT THE TORUS SURFACE

In the previous sections, we derived approximate expressions for the torus potential in the outer ($\eta^2 + \zeta^2 \geq r_0^2$) and inner ($\eta^2 + \zeta^2 \leq r_0^2$) regions. It has been shown also that the inner potential of the torus can be represented by a series in powers of η/r_0 and ζ/r_0 , and the constant, linear and quadratic terms of the series were determined analytically. To find a larger number of the series terms sufficient to represent the inner potential accurately enough, and to obtain a continuous approximate solution for the potential and its derivatives in the whole region that would satisfy the boundary conditions at the surface, we will act in the following way. Represent the inner potential of the torus as a power series⁵:

$$\begin{aligned} \phi(\eta, \zeta; r_0) = & \frac{1}{2\pi} \left(c(r_0) + \sum_{i=1} a_i(r_0) \left(\frac{\eta}{r_0} \right)^i + \right. \\ & \left. + \sum_{i=1} \sum_{j=1} t_{ij}(r_0) \left(\frac{\eta}{r_0} \right)^i \left(\frac{\zeta}{r_0} \right)^j + \sum_{j=1} b_j(r_0) \left(\frac{\zeta}{r_0} \right)^j \right), \end{aligned} \quad (36)$$

where $c(r_0)$, $a_i(r_0)$, $b_j(r_0)$, $t_{ij}(r_0)$ are unknown coefficients. Note, that the series (36) contains only the terms with even powers of ζ , because the torus potential is symmetric in ζ .

Suppose that we have an analytical expression for the torus potential $\Psi(\eta, \zeta; r_0)$ at its surface ($\eta^2 + \zeta^2 = r_0^2$). Also, write down the inner potential of the torus (36) at its surface ($\eta = r_0 \cos \theta$ and $\zeta = r_0 \sin \theta$):

$$\begin{aligned} \phi(\theta, r_0) = & \frac{1}{2\pi} \left(c + \sum_{i=1} a_i \cos^i \theta + \right. \\ & \left. + \sum_{i=1} \sum_{j=1} t_{ij} \cos^i \theta \sin^j \theta + \sum_{j=1} b_j \sin^j \theta \right). \end{aligned} \quad (37)$$

From conditions of equality of the inner and outer potential and its derivatives in coordinates at the torus surface for several angles θ_k , we obtain a system of $3k$ linear equations to determine coefficients c , a_i , b_j , t_{ij} :

$$\begin{cases} c + \sum_{i=1} a_i \cos^i \theta_k + \sum_{i=1} \sum_{j=1} t_{ij} \cos^i \theta_k \sin^j \theta_k + \\ + \sum_{j=1} b_j \sin^j \theta_k = 2\pi \Psi(\theta_k, r_0) \\ \sum_{i=1} i \cdot a_i \cos^{i-1} \theta_k + \sum_{i=1} \sum_{j=1} i \cdot t_{ij} \cos^{i-1} \theta_k \sin^j \theta_k = \\ = 2\pi r_0 \frac{\partial}{\partial \eta} \Psi(\theta_k, r_0) \\ \sum_{j=1} j \cdot b_j \sin^{j-1} \theta_k + \sum_{i=1} \sum_{j=1} j \cdot t_{ij} \cos^i \theta_k \sin^{j-1} \theta_k = \\ = 2\pi r_0 \frac{\partial}{\partial \zeta} \Psi(\theta_k, r_0) \end{cases} \quad (38)$$

⁵ We consider a dimensionless potential here. To pass to the dimensional case, (36) must be multiplied by GM/R .

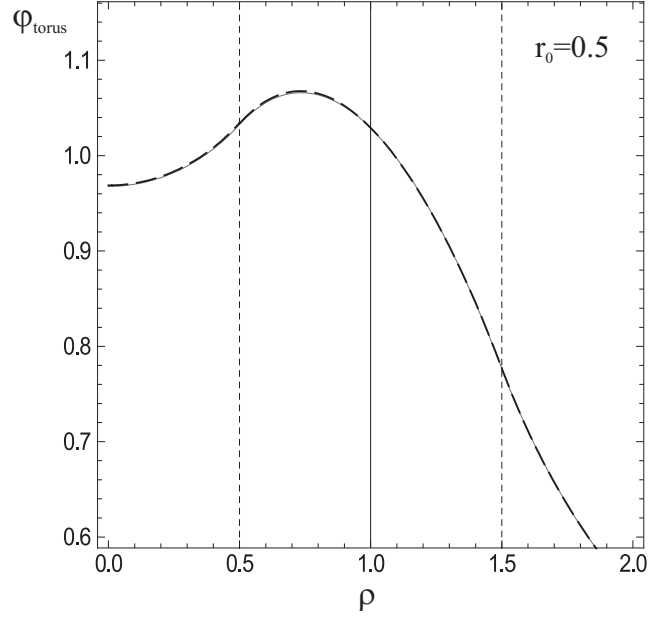


Figure 7. Dependence for the torus potential on ρ for $r_0 = 0.5$ ($\zeta = 0$) in the whole region: the potential curve calculated from the exact expression (8) is shown by a solid line; dashed line demonstrates a result of sewing together the S-approximation in the outer region (11) with the inner potential represented by the power series (37) with the coefficients found from the sewing condition (39).

Thus, if we had the analytic solution for the outer potential of the torus, we could obtain an exact expression for the inner potential as an infinite series in powers of $\cos \theta_k$, $\sin \theta_k$, using the boundary conditions and solving the system of equations (38). Since there is no analytic expression for the outer potential, we can use the above approximate expression (11) for the torus potential in the outer region (the S-approximation), and introduce designations:

$$\begin{aligned} \Phi &= \sum_k [\phi_{in}(\theta_k, r_0) - \phi_{out}(\theta_k, r_0)]^2 \\ \Phi_1 &= \sum_k \left(\frac{\partial}{\partial \eta} [\phi_{in}(\theta_k, r_0) - \phi_{out}(\theta_k, r_0)] \right)^2 \\ \Phi_2 &= \sum_k \left(\frac{\partial}{\partial \zeta} [\phi_{in}(\theta_k, r_0) - \phi_{out}(\theta_k, r_0)] \right)^2, \end{aligned}$$

where ϕ_{in} , ϕ_{out} are solutions for the inner (37) and outer (11) potentials at the torus boundary, respectively. The unknown coefficients of the series can be then determined from a condition of the minimal value of a functional:

$$F = \Phi + \Phi_1 + \Phi_2 \rightarrow \min. \quad (39)$$

The functional (39) was minimized with the least squares method, and coefficients of the series (37) were determined up to the 4-th power. The coefficients of the series are presented in Appendix (Table A1). In Fig. 7 dependence of the potential on the radial coordinate from the exact expression (8) is presented for the entire region, as well as its approximate solution obtained by sewing together the S-approximation (11) and the inner potential (37). Though the approximate solutions were obtained assuming that the torus is thin $r_0 \ll 1$, we see that the exact (8) and approximate solutions are consistent even for the torus with $r_0 = 0.5$. In Fig. 8, the equipotential curves on the plane of

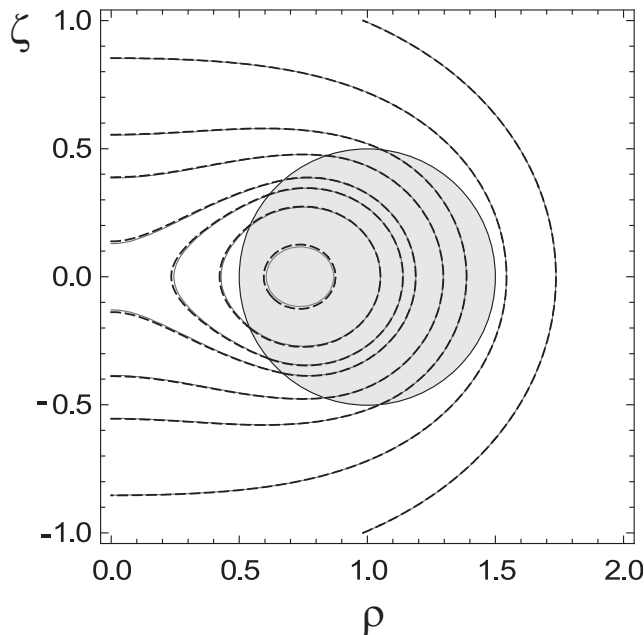


Figure 8. Equipotential curves for the torus with $r_0 = 0.5$. Solid lines are those calculated from the exact expression for the potential; dashed lines are those for the approximate formulas for the outer and inner potentials (see the figure 7 caption). The torus body cross-section is indicated as a gray circle.

the torus cross-section are shown, where a good agreement for all values of ρ , ζ is seen as well.

6 CONCLUSIONS

In the present work, the gravitational potential of a homogeneous circular torus is investigated in details. An integral expression for its potential that is valid for an arbitrary point is obtained by composing the torus of infinitely thin rings. This approach has made it possible to find an approximate expression for the outer potential of the torus (S-approximation), that has a sufficiently simple form. It is shown that the outer potential of the torus can be represented with good accuracy by a potential of an infinitely thin ring of the same mass. The dependence of the geometrical parameter r_0 appears only in the torus hole; it is taken into account in the "shifted" potential of the infinitely thin ring. These approximations are valid up to the surface of the torus.

For the inner potential, an approximate expression is found in the form of a power series to the second-order terms, where the coefficients depend only on the geometric parameter r_0 . Expressions for the potential in the center of the torus cross-section and for coordinates of the potential maximum are obtained, and the limiting passage to a cylinder potential is considered. It is shown that the inner potential of the torus can be represented as a sum of the cylinder potential and a term comprising a geometrical curvature of the torus surface. A method for determining the torus potential over the whole region is proposed that implies sewing together at the surface of the outer potential (S-approximation) with the inner potential represented by the power series. This method provided a continuous approximate solution for the potential and its derivatives, working throughout the region.

Surely, matter distribution within a torus is inhomogeneous in actual astrophysical objects, and a torus cross-section may differ from a circular one. Therefore, it is further interesting to account for inhomogeneity of matter distribution inside a torus, for difference of the torus cross-section from a circular form, and so on.

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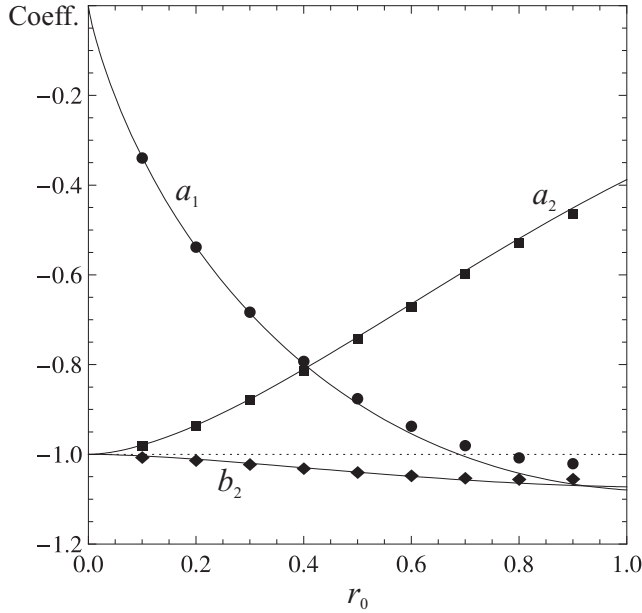


Figure A1. Dependence the first coefficients of the power series on r_0 of for the inner potential obtained with the method of sewing.

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APPENDIX A: COEFFICIENTS OF THE POWER SERIES FOR INNER POTENTIAL OF A TORUS

In Table A1, coefficients of the power series (up to the 4-th power) for the inner potential of the torus, are presented, which were calculated from the sewing condition for the torus with various values of the geometrical parameter r_0 . The analytic expression (28) was used to determine the zero-th coefficient c of the series. In Fig. A1, the linear (a_1) and quadratic (a_2, b_2) coefficients of the power series as functions of r_0 obtained analytically from (27) are shown by solid lines; the dots are the proper values of these coefficients obtained from the condition of sewing (see Table A1). The values of the analytic coefficients are seen to coincide up to $r_0 = 0.5$ with their values obtained independently with the method of sewing from (39).

Table A1. Coefficients of the power series for the inner potential of the torus for various values of r_0 , obtained with the method of sewing.

Coeff.	Geometrical parameter r_0								
	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
a_1	-0.33798	-0.53651	-0.68154	-0.79129	-0.87439	-0.93587	-0.97906	-1.00628	-1.01928
a_2	-0.98002	-0.93543	-0.87773	-0.81171	-0.74107	-0.66865	-0.59677	-0.52739	-0.46224
b_2	-1.00411	-1.01086	-1.01970	-1.02892	-1.03759	-1.04495	-1.05030	-1.05299	-1.05237
a_3	0.02392	0.04364	0.05781	0.06608	0.06853	0.06550	0.05753	0.04525	0.02938
t_{12}	0.02550	0.05329	0.08404	0.11791	0.15454	0.19323	0.23295	0.27238	0.30991
a_4	-0.00182	-0.00785	-0.01610	-0.02576	-0.03580	-0.04535	-0.05371	-0.06036	-0.06495
b_4	0.00061	0.00131	0.00308	0.00570	0.00922	0.01362	0.01880	0.02453	0.03045
t_{22}	-0.00122	-0.00812	-0.01948	-0.03681	-0.06076	-0.09157	-0.12900	-0.17213	-0.21931